MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

**O. M. BEKETOV NATIONAL UNIVERSITY
of URBAN ECONOMY in KHARKIV**

Methodological guidelines
for practical classes,
self-dependent and calculator-graphical works
on the subject

"THEORETICAL MECHANICS"
(PART 1 STATICS)

(for the first year full-time students specialty
192 – Construction and Civil Engineer)

**Kharkiv
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INTRODUCTION

Theoretical mechanics is one of the most important sciences of the physical and mathematical, which forms the scientific outlook of the engineer. Its laws are based on such general engineering disciplines as the mechanics of materials, structural mechanics, applied mechanics, machinery, the theory of machines and mechanisms, hydraulics, etc. Therefore, the study of theoretical mechanics is necessary both for understanding these disciplines, and for the scientific interpretation of nature phenomena.

Definition:
Theoretical mechanics is the science of the most general laws of mechanical motion and equilibrium of material bodies and the resulting interactions between them.

STATICS

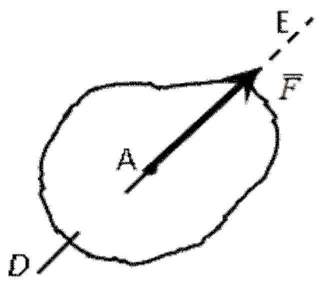
The statics is a part of theoretical mechanics studies the methods of converting the given systems of forces into other, equivalent ones, as well as the conditions for the equilibrium of rigid body under the action of the force system.

Absolutely rigid body is a material body in which the distance between any two points does not change.

Material point – the simplest model of the material body, which dimensions under the conditions of this problem, can be neglected.

Mechanical system – a set of interconnected material points, the position and displacement one depends on the position and disposition of all others.

Force is a measure of the mechanical interaction of two bodies. Force is a vector value determined by the magnitude, action and direction line, and application point.



Picture 1

The unit of force measurement is 1 newton ($1\ N = 1\ kg\ m / s^2$), or 1 kilogram of force (1 kG), 1 kG ≈ 9,8 N.
Line of force – line *DE*, along which force is direct.
System of forces – a set of forces acting on body.

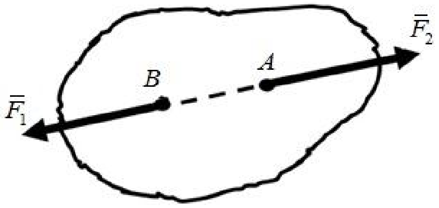
Equivalent systems of forces – a system of forces, under the influence of which a rigid body is in the same state (equilibrium or motion).

A balanced system of forces (or equivalent to zero) – a system of forces under which a rigid body is in equilibrium.

The resultant system of forces – force equivalent to the given force system.

Statics is based on six axioms:

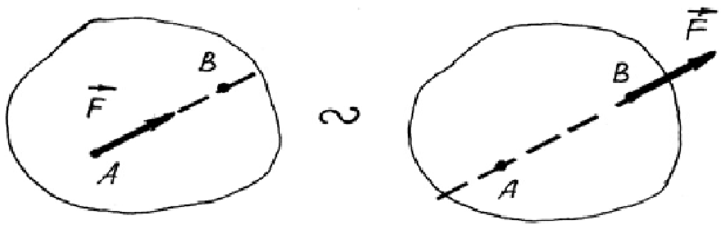
1. A rigid body acted upon by two forces is in a state of static equilibrium if and only if the two forces are of the same intensity, lie along the same line of action, and are oriented in opposite directions along the line.



$$\overline{F}_1 = -\overline{F}_2$$

Picture 2

2. If a system of two forces in equilibrium is added to or extracted from a given system of forces, in this the way the system of forces acting on a rigid body unchanged.



Picture 3

Consequence of axioms 1 and 2: The effect of force on absolutely rigid body does not change if you move the force along the line of action to another point in the body.

3. The resultant of two forces acting at the same material point is equal to the vector sum of the two forces. The line of the resulting force's action passes the material point. This axiom obeys the principle of vector summation.

4. Two interacting bodies react on each other with two forces of equal intensity, and along the same line of action, but in opposite directions along the line. This axiom is also known as principle of action and reaction.

5. A non-free material body can be treated as free if the constrains are replaced by their reactions.

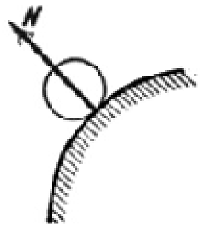
6. If a deformable body is in a state of static equilibrium, it would also be in static equilibrium if the body were rigid. This axiom is also known as the principle of solidification.

Free body is a body, the displacement of which is not restricted by any other body.

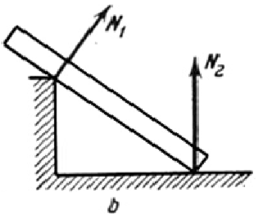
Constrains are the bodies which prevent the motion of the body.

The ***constrain reaction*** is the force with which the constrain acts on the body, the movement of which it restricts.

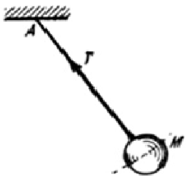
Constrains and direction of their reactions:



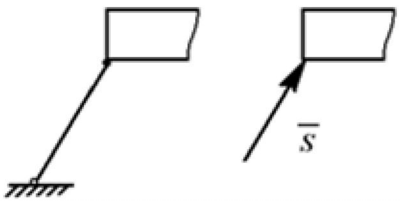
Smooth surfaces contact. Contact force N is compressive and directed along the normal surface. (One force with known line of action).



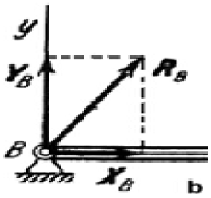
Angle. Reaction N_1, N_2 is directed perpendicular to the plane.



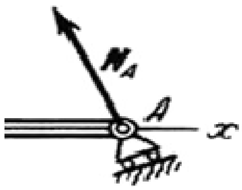
Rigid weightless link, weightless flexible, non-stretchable cord (String). Rigid link reaction is directed along the line connecting the centers of the link.



Uniform Rod. The reaction S of a rod is directed along its axis.



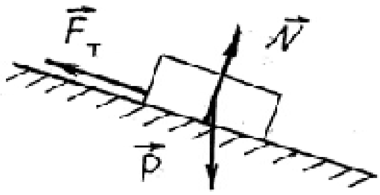
Pinned joint B. Prevents the motion of the beam end in two mutually perpendicular directions: along axes x and y.



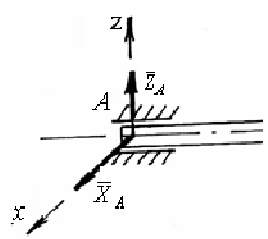
Sliding joint or roller support A. Sliding joint prevents motion of the beam end A in direction that is perpendicular to the plane where the joint is installed.

Fixed End Support

Fixed end support, clamped joint - This support prevents rotation, horizontal and vertical motion. The support will have a moment, and a horizontal and vertical force component.

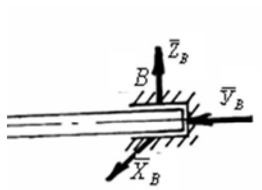


Force of Sliding friction F_{fr}. Reaction support can be decomposed into two components: the force normal to the surface N and force tangent to the surface resistance F_{fr} (friction force). The friction force is: $F_{fr} \leq fN$,
where f - friction coefficient, N – normal reaction.



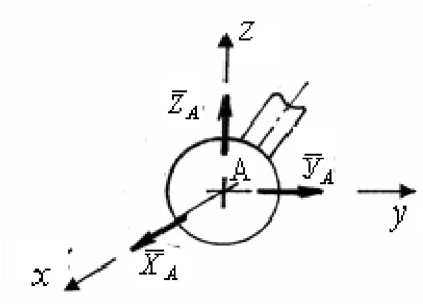
Bearing support. Cylindrical *bearing* does not prevent the displacement of the body in only one direction along the cylinder axis. The *bearing support* reaction has two components in two mutually perpendicular directions.

$$\overline{R}_A = \overline{X}_A + \overline{Z}_A, \qquad R_A = \sqrt{X_A^2 + Z_A^2}$$



$$\overline{R}_B = \overline{X}_B + \overline{Y}_B + \overline{Z}_B$$

Trust bearing support. The cylindrical *trust bearing* doesn't prevent the movement of the body in only one direction along the axis of the cylinder and prevents the displacement in the opposite direction. The *trust bearing support* reaction has three components in three mutually perpendicular directions.

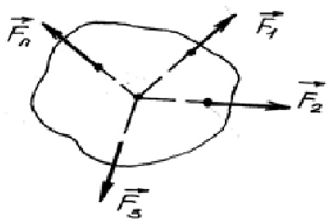


Ball-and-socked joint. In the case of a *Ball-and-socked joint*, the body can rotate around the center of the hinge, but it is not able to move in any direction. The *Ball-and-socked joint* reaction has three components in three mutually perpendicular directions:

$$\overline{R}_A = \overline{X}_A + \overline{Y}_A + \overline{Z}_A$$

Axiom 6 of contact which permits to reduce the problems of equilibrium of *constrained* bodies to study of free ones: *A non-free material body can be considered as free if the constrains are replaced by their reactions.*

1 CONCURRENT FORCE SYSTEM

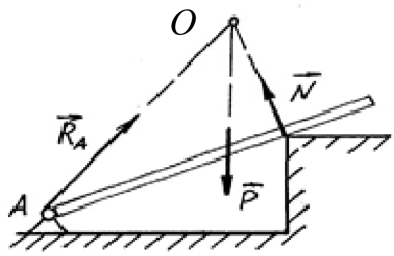


Picture 4

Concurrent force system is the system of forces, which lines of action intersect at one point. Point O is the point of this forces intersection.

The Theorem of Three Forces: *if a free rigid body is in equilibrium under the action of three nonparallel coplanar forces, which lines of action intersect at one point.*

With this theorem, you can define a line of action, for example, a reaction R_A of a pinned joint, considering the lines of force action of weight P and reaction N are known.



Picture 5

Theorem (*about resultant concurrent force system*): A concurrent force system is equivalent to a single force R (*resultant*), which equals the vector sum of the system forces and applied to the point where all the lines of forces action are intersect.

The conditions of ***concurrent force system*** *equilibrium*:

– for the concurrent force system to be in equilibrium it is necessary and sufficient for the *resultant* of the forces to be zero (geometrical conditions):

$$\overline{R} = 0$$

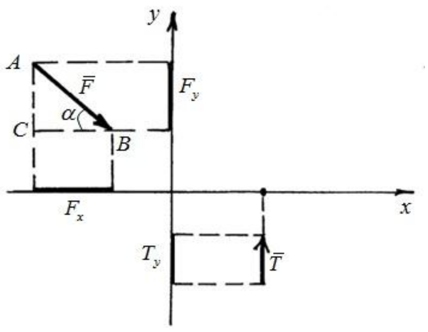
– the sums of the projections of all forces on each of the coordinate axes must be zero (analytically conditions):

$$\sum_{i=1}^{n} F_{ix} = 0; \qquad \sum_{i=1}^{n} F_{iy} = 0; \qquad \sum_{i=1}^{n} F_{iz} = 0.$$

– if all the forces of concurrent force system are lied in a plane, such as xOy, then their equilibrium must be

$$\sum_{i=1}^{n} F_{ix} = 0; \qquad \sum_{i=1}^{n} F_{i-} = 0$$

2 PROJECTION OF FORCE F ON THE AXIS

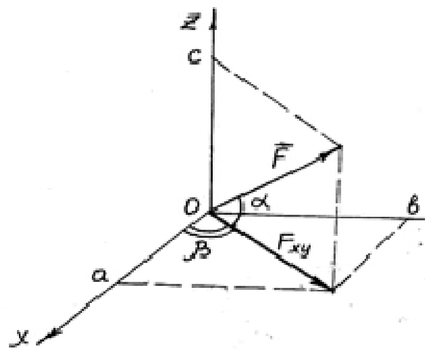


Picture 6

Projection of force on the axis is an algebraic quantity equal to the length of the segment between the projections of the beginning and the end of the force on this axis. The projection has the sign "+" if the force vector is inclined toward the positive direction of the axis and the sign "-" if the direction is negative.

The magnitude of the projection of force is determined by the solution of the triangle ABC, which is formed by the initial force and the lines parallel to the axes of the coordinates.

If the force is perpendicular to the axis, then its projection to this axis is zero.



Picture 7

Therefore $F_x = F \cdot \cos\alpha$, $F_y = - F \cdot \sin\alpha$,

$T_x = 0$, $T_y = T$.

If the force is in space, then its projections are defined as follows:

$$F_x = F_{xy} \cdot \cos\beta = F\cos\alpha\cos\beta,$$

$$F_y = F_{xy} \cdot \sin\beta = F\cos\alpha\sin\beta,$$

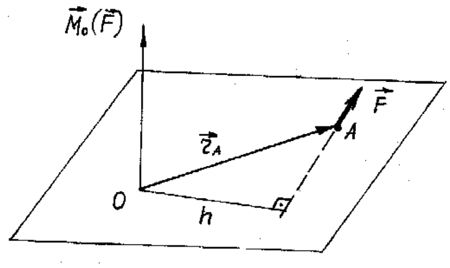
$$F_z = F\sin\alpha.$$

3 MOMENTS OF FORCE AND COUPLE FORCES

3.1 Moment of force about the point

Arm h forces *F* about the point *O* is called the shortest distance from point O to the line of force (perpendicular), conducted from point *O* to the line of force.

Moment magnitude is $M_0(\overline{F}) = F \cdot r_A \cdot \sin\alpha = F \cdot h$.



Picture 8

The moment of force about the point O is a vector which magnitude is equal to the product of force on the arm and applied to this point *O*. It is directed perpendicular to a plane passing through the point *O* and of force action line, in the direction from which the rotation of force about the point is visible counter clockwise.

The vector of the force moment can presented as a vector product of the vector-position of the point of force application to the force vector:

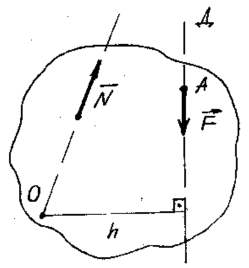
$$\overline{M}_0(\overline{F}) = \overline{r}_A \cdot \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \overline{i}\,(yF_z - F_y) + \overline{j}(zF_x - xF_z) + \overline{k}(xF_y - yF_x)$$

where x, y, z - Cartesian coordinates of point A of applied force; F_x, F_y, F_z - projection of force to the coordinate axis; $\overline{i}, \overline{j}, \overline{k}$ - orts axes x, y, z.

When solving problems in a plane, the definition of an *algebraic moment of force* about a point is used.

The rule for determining the algebraic moment of force ***F*** about the point:

1. Draw a line of force (direct *DE*).
2. From the chosen point *O*, draw a perpendicular to the force line (its length ***h*** is the arm of force).



Picture 9

3. Write the product of the force magnitude on the ***arm*** $(F \cdot h)$.
4. Use the "+" sign if the force tends to rotate the arm about the point counter-clockwise movement, and the sign "-" – if it rotates clockwise.

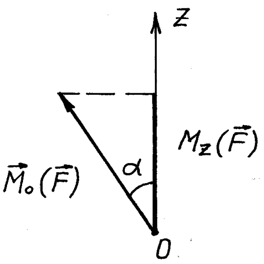
$$M_0(\overline{F}) = \pm F \cdot h.$$

The algebraic moment of force about a point is *zero* if the line of force passes through this point. Then the arm ***h* = *0***, and the moment

$$M_0(\overline{N}) = F \cdot 0 = 0.$$

3.2 Moment of force about the axis

The moment of force about the *z*-axis is the projection the force moment about the point lying on the axis on this axis:

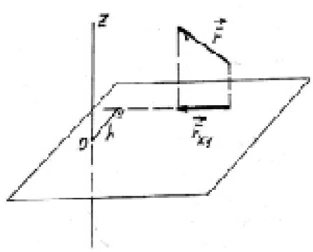


Picture 10

$$M_z(\overline{F}) = \left[\overline{M}_0(\overline{F})\right]_z = M_0(\overline{F}) \cdot \cos\alpha$$

The moment of force about the axis characterizes the rotational force of the force around this axis.

The rule for determining the algebraic moment of force F about the z-axis:



1. Draw a plane perpendicular to the z axis and find the point O of the axis intersection with the plane.
2. Find the projection force on the drawn plane (vector - projection of force on a plane).

Picture 11

3. Find the moment of the received force $\overline{F}_{xy}$ about to the point of intersection of the axis with the plane *Oxy*: $M_0(\overline{F}_{xy}) = \pm F_{xy} \cdot h$. Denote the sign "+" if a positive projection of the z-axis shows that the force projection tends to rotate the plane around z axis counter clockwise and the sign "–" if rotates clockwise.
4. The moment of force $\overline{F}$ about the z-axis is defined as:

$$M_z(\overline{F}) = M_0(\overline{F}_{xy}) = \pm F_{xy} \cdot h.$$

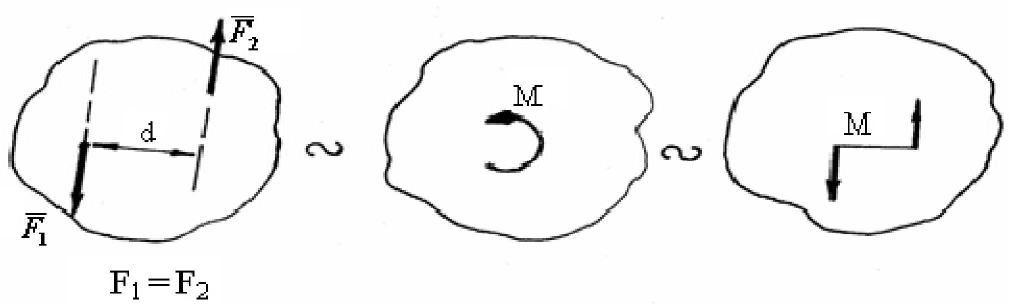
The moment of force about the axis is *zero* if:

a) the force is parallel to the axis (in this case, the projection of force on the plane $F_{xy} = 0$);

b) the line of force intersects the axis (with this arm $h=0$).

3.3 Moment of a couple

A force couple *is a system of two parallel forces of the same magnitude and opposite direction*. A force system constituting a couple is not in equilibrium. This conclusion follows from the first axiom of statics.



Picture 12

The arm couple d is the shortest line between the forces lines the couple consist of.

Denoting the moment of a couple by the symbol ***M***, we have:

$$M = \pm F \times d\ .$$

Magnitude the moment of a couple M

$$M(\overline{F}_1, \overline{F}_2) = F_1 \cdot d = F_2 \cdot d.$$

The moment of a couple M is said to be positive if the action of the couple tends to turn a body counterclockwise, and negative if clockwise.

Properties of a ***couple***:

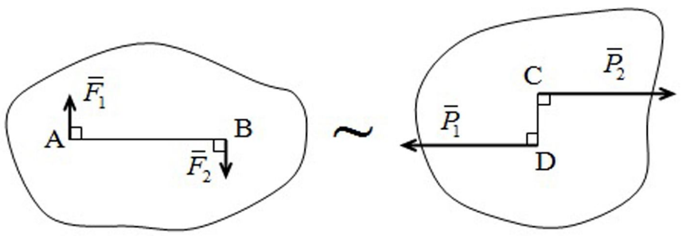
1. A couple has no resultant. Therefore, a couple cannot be changed or balanced by one force; it can be balanced by another pair.

2. The geometric sum of the forces moments that compose a couple about any point of *O* does not depend on the choice of this point and is equal to the moment of a couple:

$$\overline{M}_0(\overline{F}_1) + \overline{M}_0(\overline{F}_2) = \overline{M}(\overline{F}_1, \overline{F}_2).$$

3. Two couples are equivalent if their moments are geometrically equivalent.

The corollary of this property is that a couple acting on a rigid body can be moved in the plane of its action, or in parallel plane, it is possible to change the values of forces or a couple of arms, but save the value of couple and direction of rotation:



if $F_1 \cdot AB = P_1 \cdot CD$.

Picture 13

4. The system of same couples arbitrarily located in space is equivalent to one couple, the moment of which is equal to the geometric sum of the moments of the composed couples:

$$\overline{M} = \sum_{i=1}^{n} \overline{M}_i .$$

Equilibrium condition of the system of couples:

Couples of forces arbitrarily located in space, are in equilibrium, if the geometric sum of their moments is zero:

$$\sum_{i=1}^{n} \overline{M}_i = 0.$$

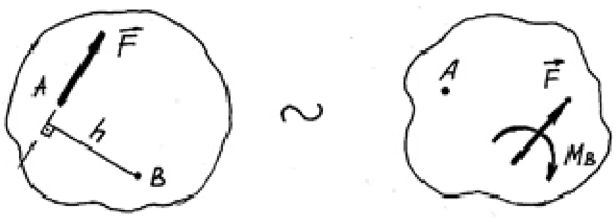
4 THREE-DIMENSIONAL FORCE SYSTEM

Three-dimensional force system is a force system, arbitrarily located in space.

According to the axiom 2 the statics, the force, applied to the body, can be moved along its line of action to any other of its point. In this case the state of the body does not change.

In a number of practical problems of equilibrium connected with the simplification of a given system of forces, there is often takes place the need to move force to a given center parallel to itself. The parallel transposition of force leads to a change in the system of force factors as compared with its initial mechanical state.

Lemma Poinsot's: *the force applied at the point O of a rigid body is equivalent to the same force applied at another point of the body and a couple of forces with the moment equal to the moment of the original force about the new center.*



Magnitude of moment

$$M = M_B(\overline{F}) = F \cdot h$$

Picture 14

The principal vector of the system is the quantity F_O, *which is the geometrical sum of all the forces of the given system.*

$$\overline{F} = \sum_{i=1}^{n} \overline{F}_i$$

Magnitude of *the principal vector* is determined by formula:

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2},$$

where F_x, F_y, F_z are projections of the principal vector on axis:

$$F_x = \sum_{i=1}^{n} F_{ix}; \quad F_y = \sum_{i=1}^{n} F_{iy}; \quad F_z = \sum_{i=1}^{n} F_{iz}.$$

The direction *the principal vector* F_O is determined by cosines directions:

$$\cos\left(\widehat{x, \overline{F}}\right) = \frac{F_x}{F}; \quad \cos\left(\widehat{y, \overline{F}}\right) = \frac{F_y}{F}; \quad \cos\left(\widehat{z, \overline{F}}\right) = \frac{F_z}{F}.$$

The principal moment *of the force system about center O is the quantity* M_O, *which is the sum of the moments of all the forces of the system about center O.*

$$\overline{M}_0 = \sum_{i=1}^{n} \overline{M}_0(\overline{F}_i).$$

Magnitude of *the principal moment* of the force system about center *O* equals*:*

$$M_0 = \sqrt{M_{0x}^2 + M_{0y}^2 + M_{0z}^2},$$

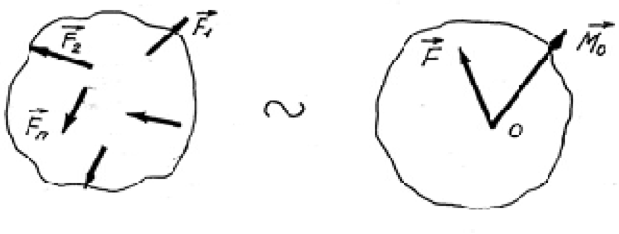
Where M_{0x}, M_{0y}, M_{0z} are projections of *the principal moment* on axis:

$$M_{0x}=\sum_{i=1}^{n}M_x(\overline{F}_i);\quad M_{0y}=\sum_{i=1}^{n}M_y(\overline{F}_i);\quad M_{0z}=\sum_{i=1}^{n}M_z(\overline{F}_i).$$

The direction of *the principal moment* M_O is determined by cosines directions:

$$\cos\left(x,\widehat{\overline{M}_0}\right)=\frac{M_{0x}}{M_0};\quad \cos\left(y,\widehat{\overline{M}_0}\right)=\frac{M_{0y}}{M_0};\quad \cos\left(z,\widehat{\overline{M}_0}\right)=\frac{M_{0z}}{M_0}.$$

The Main Theorem of Statics: *General force system applied to a rigid body can be reduced to an equivalent force-couple system acting at a given point O. The force is equal to the total vector* $\overline{F}_O$. *The couple has a vector moment equal to the total moment* $\overline{M}_O$ *of original force system about the point O.*



where $\overline{F}_O=\sum_{i=1}^{n}\overline{F}_i$,

$$\overline{M}_O=\sum_{i=1}^{n}\overline{M}_O(\overline{F}_i).$$

Picture 15

Separate cases of reduction force system

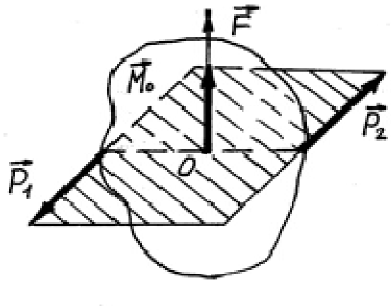
Let us consider different combinations of *the total vector* and *total moment* values as special cases of the general force system:

1) If $F_O = 0$ and $M_O = 0$ the system is ***balanced*;**

2) If $F_O = 0$ and $M_O \neq 0$, the system can reduced to a couple of moment $M_0=\sum M_k=\sum M_0\left(F_k\right)$ equal to ***the principal moment*** of the system. In this case, the magnitude of M_O does not depend on the location of the center *O*;

3) If $F_O \neq 0$ and $M_O = 0$. The initial force system is equivalent to a single force is a ***resultant R***, which passes through center *O*;

4) If $F_O \neq 0$ and $M_O \neq 0$. We shall consider two special cases:



$\overline{F} \neq 0$, (i. e. $\overline{M}_0 \neq 0$ and $\overline{M}_0 \perp \overline{F}$) $(\overline{F} \cdot \overline{M}_0) \neq 0$ – system reduces to ***wrench,*** which is consists force ***F*** and couple $\boldsymbol{M_o}$. If $M_O \| F$, that axis of wrench pass through the center of reduction, if $M_O \nparallel F$ that axis of wrench doesn't pass through the center.

Picture 16

Conditions for the Equilibrium of a Coplanar Force System. For any given coplanar force, system to be in equilibrium it is necessary and sufficient for the following two conditions to be satisfied simultaneously:

$$\overline{F}_O = 0 \text{ and } \overline{M}_O = 0,$$

where *O* is any point in a given plane.

Analytical conditions: the sum of the projections of all forces on each of the three coordinate axes and the sum of their moments about these axes was equal to zero

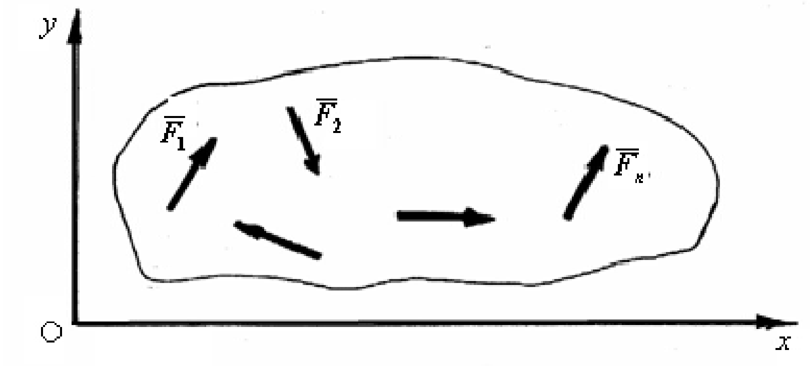
$$\sum_{i=1}^{n} F_{ix} = 0; \quad \sum_{i=1}^{n} M_x(\overline{F}_i) = 0;$$
$$\sum_{i=1}^{n} F_{iy} = 0; \quad \sum_{i=1}^{n} M_y(\overline{F}_i) = 0;$$
$$\sum_{i=1}^{n} F_{iz} = 0; \quad \sum_{i=1}^{n} M_z(\overline{F}_i) = 0.$$

Conditions of equivalence of two force systems: The two force systems $\{\overline{F}_1,...,\overline{F}_n\}$ and $\{\overline{P}_1,...,\overline{P}_m\}$ are *statically equivalent*, if their *principle vectors* $(\overline{F}$ and $\overline{P})$ and the *principle moments* $(\overline{M}_0^F$ i $\overline{M}_0^P)$ about any center are equal:

$$\overline{F} = \overline{P},$$
$$\overline{M}_0(\overline{F}) = \overline{M}_0(\overline{P}).$$

5 COPLANAR FORCE SYSTEM

The ***coplanar force system*** is a system of forces which lines of action are located in one plane.



Picture 17

Reduction of coplanar system of forces: This system of forces is reduced to a force equals the total vector of the forces system and to a couple of forces which moment equals the total moment of the system about the chosen reduction center O.

$$\overline{F}_0 = \sum_{i=1}^{n} \overline{F}_i \ , \ \ \overline{M}_0 = \sum_{i=1}^{n} \overline{M}_0\left(\overline{F}_i\right)$$

Conditions for the Equilibrium of a Coplanar Force System. For any given coplanar force, system to be in equilibrium it is necessary and sufficient for the following two conditions to be satisfied simultaneously (*geometrical conditions*):

$$\overline{F}_0 = 0 \text{ and } \overline{M}_0 = 0,$$

The magnitude of $\overline{F}_O$ and $\overline{M}_O$ are determined by the equations:

$$F_0 = \sqrt{F_x^2 + F_y^2 + F_z^2} \ , \quad M_0 = \sum M_0\left(F_k\right),$$

where $F_x = \sum F_{kx}$ and $F_y = \sum F_{ky}$. But F_0 can be zero only if both $F_x = 0$ and $F_y = 0$.

Three different forms of the analytical conditions of equilibrium.

1. ***The basic form of the equations***. Conditions of equilibrium will be satisfied if

$$\sum F_{kx} = 0, \sum F_{ky} = 0, \ \sum_{i=1}^{n} M_0(\overline{F}_i) = 0. \qquad (1)$$

For any given coplanar force system to be in equilibrium it is necessary and sufficient for the sums of the projections of all the forces on each of the two coordinate axes and the sum of the moments of all the forces about any point in the plane to be separately zero.

2. ***The second form of the equations***. *For any given coplanar force system to be in equilibrium it is necessary and sufficient for the sums of the moments of all the forces about any two points A and B , and the sum of the projections of all the forces on any axis Ox not perpendicular to AB to be separately zero:*

$$\sum_{i=1}^{n} F_{ix} = 0; \qquad \sum_{i=1}^{n} M_A(\overline{F_i}) = 0; \qquad \sum_{i=1}^{n} M_B(\overline{F_i}) = 0.$$

3. ***The third form of the equations***. *For any given coplanar force's system to be in equilibrium it is necessary and sufficient for the sums of the moment of all the forces about any three non-collinear points to be* zero:

$$\sum_{i=1}^{n} M_A(\overline{F_i}) = 0; \qquad \sum_{i=1}^{n} M_B(\overline{F_i}) = 0; \qquad \sum_{i=1}^{n} M_C(\overline{F_i}) = 0\,. \tag{3}$$

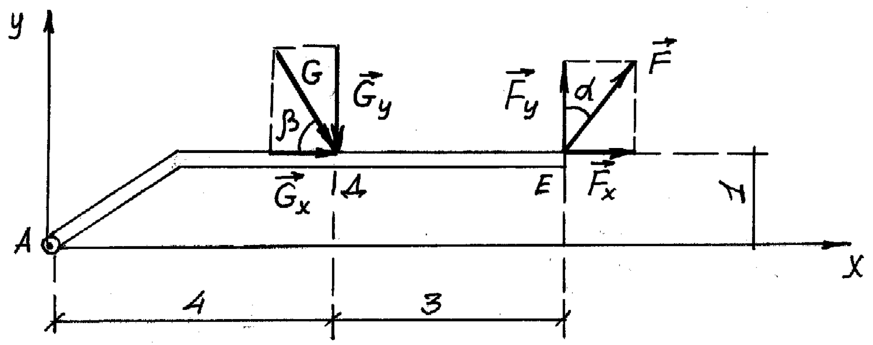
(Points ***A, B, C*** belong to different lines).

Varignon's theorem. *If a force system acting upon a rigid body is equivalent to the resultant force $\overline{R}$, the resultant force moment about any chosen center O is equal to the sum of the moments of the system forces about the same center.*

$$\overline{M}_0(\overline{R}) = \sum_{i=1}^{n} \overline{M}_0(\overline{F_i})$$

This theorem is use for finding an algebraic moment of force about a point, decompose the force into components, parallel to the axes of coordinates:

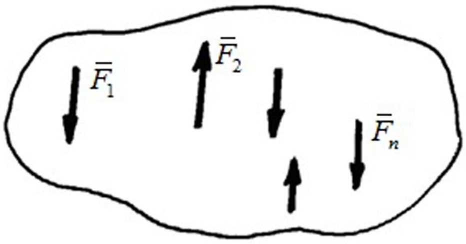
$$M_A(\overline{G}) = M_A(G_x) + M_A(G_y) = -G_x \cdot 1 - G_y \cdot 4,$$
$$M_A(\overline{F}) = M_A(F_x) + M_A(F_y) = -F_x \cdot 1 + F_y \cdot 7,$$



Picture 18

where magnitude of components
$G_x = G \cdot \cos\beta,$
$G_y = G \cdot \sin\beta\,,$
$F_x = F \cdot \sin\alpha,$
$F_y = F \cdot \cos\alpha.$

6 PLANE SYSTEM OF PARALLEL FORCES. DISTRIBUTED FORCES



Picture 19

The *system of parallel forces* is a system of forces which lines of action are parallel to each other.

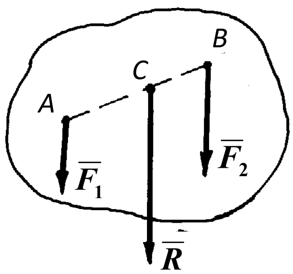
Condition of Equilibrium a plane system of parallel forces: it is necessary and sufficient enough to be algebraic sum the forces projections on the axis parallel to the forces, and the algebraic sum of forces moments about any point on the plane equal to zero:

$$\sum_{i=1}^{n} F_{iy} = 0; \qquad \sum_{i=1}^{n} M_0(\overline{F}_i) = 0$$

(if all forces are parallel to the ***Oy*** axis).

Addition two parallel forces:

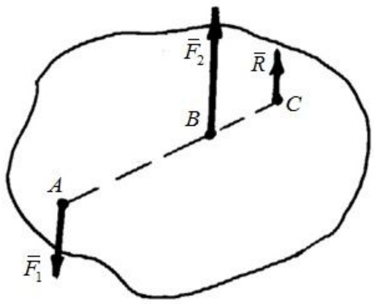
1. The forces are directed in one direction:



$$R = F_1 + F_2,$$
$$F_1 \cdot AC = F_2 \cdot BC.$$

Picture 20

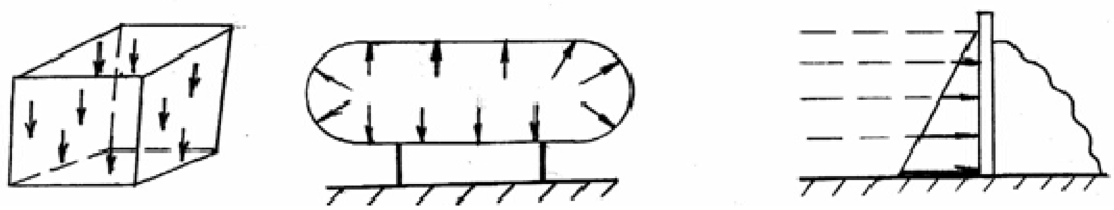
2. The forces are directed in opposite direction $(F_2 > F_1)$:



$$R = F_2 - F_1,$$
$$F_1 \cdot AC = F_2 \cdot BC.$$

Picture 21

Distributed forces (distributed load) - a system of parallel forces acting on each point of volume, surface or line.

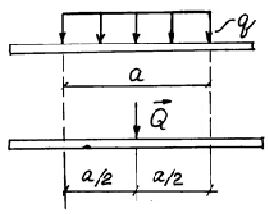


Picture 22

The system of distributed forces (forces distributed along the line) is characterized by its intensity ***q***, that is, by force per unit length: ***[q] = N/m.***

Simple examples of distributed forces:

1) uniformly distributed load: $q = const$



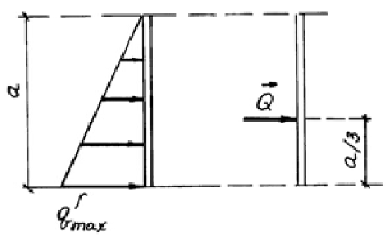
Picture 23

Magnitude of resultant force $\overline{Q}$ equals

$$Q = q \cdot a$$

and applied in the middle of the segment on which the distributed forces acted;

2) distributed load with linear variation:



Picture 24

Magnitude of resultant force $\overline{Q}$ equals

$$Q = \frac{1}{2} q_{\max} \cdot a$$

and applied on distance *a/3* of the end segment on which the intensity is maximum.

7 METHODS OF SOLVING PROBLEMS ON THE BODIES SYSTEM OF EQUILIBRIUM

If the construction consists of some rigid bodies, interconnected by constrains (composite structure), then one can solve the problem in one of two ways:

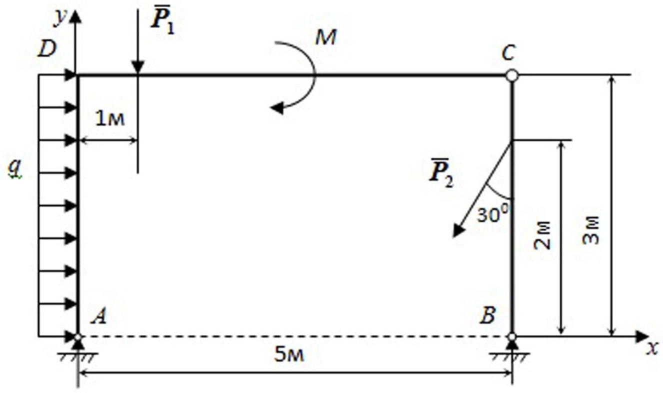
1) examine the equilibrium of the whole structure and, in addition, the equilibrium of one or more separate solids constituting the structure;

2) the initial structure at ones cut into separate rigid bodies and examine the equilibrium of each of them separately.

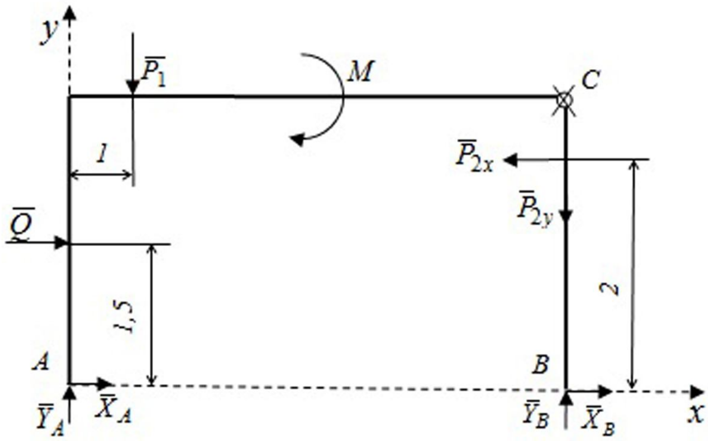
Problem 1. Two weightless rods *ADC* and *BC* are connected with each other by a hinge *C* and fixed by the pined joints *A* and *B*. Forces $P_1 = 10\ kN$, $P_2 = 20\ kN$, distributed load $q = 4\ kN / m$ and a couple with a moment $M = 50\ kNm$ are acted to the construction. Dimensions are given on the given scheme. To determine the reactions of the constraints *A* and *B*, as well as the pressure in the intermediate hinge *C* of the composite design.

Solution: this problem is solving to first way, examine the equilibrium of the whole structure and examine rod *CB* separately. Draw the free body diagram: cut constraints and change them by reactions $\overline{X}_A, \overline{Y}_A, \overline{X}_B, \overline{Y}_B$, conversion the distributed load resulting force $Q = 3q$ applied to the center of segment *AD*. Draw the coordinate axis.

The hinge C is considered to be fixed using the axiom 5 of solidification.



Picture 25



Picture 26

Determine value of forces $\overline{Q}, \overline{P}_{2x}$ and $\overline{P}_{2y}$

$$Q = 3q = 3 \cdot 4 = 12 \ kN,$$

$$P_{2x} = P \cdot \sin 30^\circ,$$

$$P_{2y} = P \cdot \cos 30^\circ.$$

whole structure:

$$\sum_{i=1}^{n} F_{ix} = X_A + X_B + Q - P_{2x} = 0, \qquad (1)$$

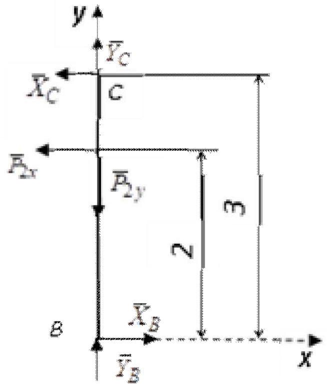
$$\sum_{i=1}^{n} F_{iy} = Y_A + Y_B - P_1 - P_{2y} = 0, \qquad (2)$$

$$\sum_{i=1}^{n} M_B(F_i) = -Y_A \cdot 5 - Q \cdot 1{,}5 + P_1 \cdot 4 - M + P_{2x} \cdot 2 = 0. \qquad (3)$$

Then cut the construction onto compose elements and examine rod *BC* separately. Action eliminated construction *ADC* replaced the reactions $\overline{X}_C, \overline{Y}_C$ into pin *C*.

Note. Directions of coordinate axes on both free body diagrams must coincide.

Compose equations of equilibrium for rod BC:



Picture 27

$$\sum_{i=1}^{n} F_{ix} = X_B - X_C - P_{2x} = 0, \tag{4}$$

$$\sum_{i=1}^{n} F_{iy} = Y_B + Y_C - P_{2y} = 0, \tag{5}$$

$$\sum_{i=1}^{n} M_C(F_i) = X_B \cdot 3 - P_{2x} \cdot 1 = 0. \tag{6}$$

In accordance with the considered free body schemes, we have six unknown reactions of the constrain $\overline{X}_A, \overline{Y}_A, \overline{X}_B, \overline{Y}_B$ and reactions $\overline{X}_C$, $\overline{Y}_C$ into pin C. Determine them from this equations.

From the third equation determine

$$Y_A = \frac{P_1 \cdot 4 + P_2 \sin 30° \cdot 2 - M - Q \cdot 1{,}5}{5} = \frac{10 \cdot 4 + 20 \sin 30° \cdot 2 - 50 - 12 \cdot 1{,}5}{5} = -1{,}6\,kN$$

from the second equation we get

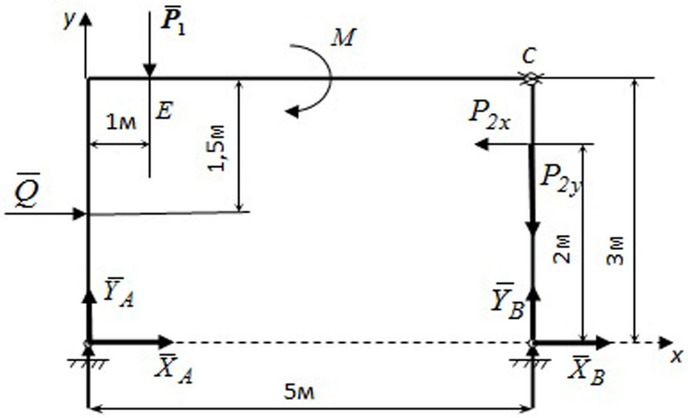
$$Y_B = -Y_A + P_1 + P_{2y} = -(-1{,}6) + 10 + 20\cos 30° = 28{,}92\ kN.$$

from the six equation we get $X_B = \dfrac{P_2 \sin 30°}{3} = \dfrac{20 \cdot \sin 30°}{3} = 3{,}33\ kN$,

from the fifth get $Y_C = P_2 \cos 30° - Y_B = 20 \cdot \cos 30° - 28{,}92 = -11{,}6\ kN$,

from forth $-$ $X_C = X_B - P_2 \sin 30° = 3{,}33 - 20 \cdot \sin 30° = -6{,}67\ kN$,

from first $-$ $X_A = -X_B - Q + P_2 \sin 30° = -3{,}33 - 12 + 20 \cdot \sin 30° = -5{,}33\ kN$.



Picture 28

Values $\overline{X}_A, \overline{Y}_A, \overline{X}_c, \overline{Y}_c$ is less than zero, its means that reactions directed oppositely shown reactions in the picture really.

Structure as a whole and compose equation of moments about the point through which the lines of action of these reactions do not pass. For example, with about the point E. The hinge C considered solidifying.

Checking:

$$\sum_{i=1}^{n} M_E(\overline{F}_i) = X_A \cdot 3 - Y_A \cdot 1 + Q \cdot 1{,}5 - M - P_{2x} \cdot 1 - P_{2y} \cdot 4 + Y_B \cdot 4 + X_B \cdot 3 =$$

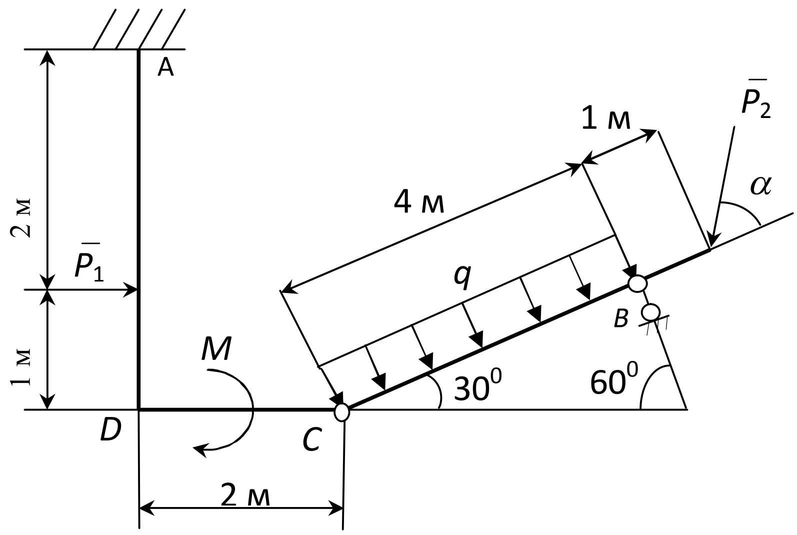
$$= -5{,}33 \cdot 3 - (-1{,}6) \cdot 1 + 12 \cdot 1{,}5 - 50 - 20\sin 30° \cdot 1 - 20\cos 30° \cdot 4 + 28{,}92 \cdot 4 +$$

$$+ 3{,}33 \cdot 3 = -15{,}99 + 8 + 18 + 40 - 50 - 10 + 9{,}9 = -145{,}27 + 145{,}27 = 0.$$

In this case, the sum of the moments of forces about the point E equals zero means determined reactions is correctly.

Problem 2. Determine reactions of constraints A and B and pressure in the intermediate pin C composite structure in which acted forces $P_1 = 6$ *kN*, $P_2 = 10$ *kN* , distributed load q = 1,4 *kN/m* and couple of forces M = 15 *kNm*. Dimensions take from given scheme, angle $\alpha = 60^{\text{o}}$.

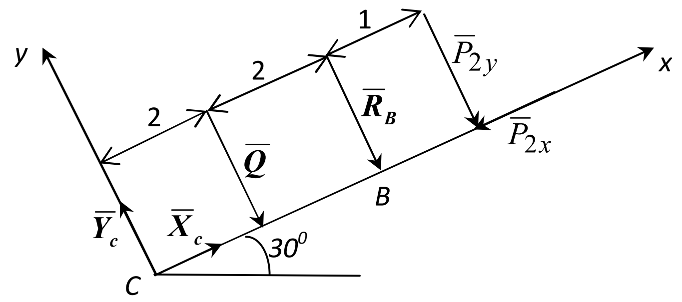
Solution. This problem will solve the second way: will consider equilibrium rods of structure *ADC* and *BC* separately. It should in mind that according to the axiom 4 of the reaction $\overline{X}_C, \overline{Y}_C$ and $\overline{X}'_C, \overline{Y}'_C$ in pin C satisfy the following equations: $X_C = X'_C, Y_C = Y'_C$. Distributed load replace on the resultant force Q = 4q, which applied into middle of segment *CB*.



Picture 29

Determine the forces $\overline{P}_{2x}, \overline{P}_{2y}, \overline{Q}$, acted into the rod *CB*:

$$P_{2x} = P_2 \cdot \cos 60° = 10 \cdot \frac{1}{2} = 5\ kN,\ P_{2y} = P_2 \cdot \sin 60° = 10 \cdot \frac{\sqrt{3}}{2} = 8{,}7\ \ kN,$$



Picture 30

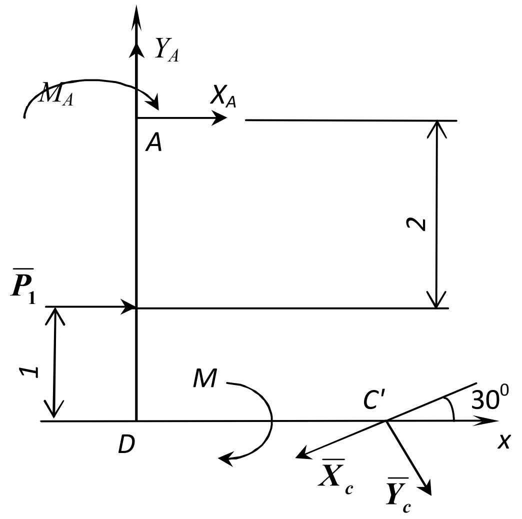
$Q = 4q = 5{,}6\ kN$.
Compose equation of equilibrium for rod *BC:*

$$\sum_{i=1}^{n} F_{ix} = X_C - P_{2x} = 0, \tag{1}$$

$$\sum_{i=1}^{n} F_{iy} = Y_C - Q + R_B - P_{2y} = 0, \tag{2}$$

$$\sum_{i=1}^{n} M_C = -Q \cdot 2 + R_B \cdot 4 - P_{2y} \cdot 5 = 0. \tag{3}$$

Compose equation of equilibrium for rod *ADC*:



Picture 31

$$\sum_{i=1}^{n} F_{ix} = X_A + P_1 - X_C \cos 30° + Y_C \sin 30° = 0, \tag{4}$$

$$\sum_{i=1}^{n} F_{iy} = Y_A - Y_C \sin 60° - X_C \sin 30° = 0, \tag{5}$$

$$\sum_{i=1}^{n} M_A = -M_A + P_1 \cdot 2 - X_C \cos 30° \cdot 3 + Y_C \cos 60° \cdot 3 - M - X_C \sin 30° \cdot 2 - Y_C \sin 60° \cdot 2 = 0. \tag{6}$$

From the six equations we define unknown reactions: $\overline{X}_A, \overline{Y}_A, M_A, \overline{X}_C, \overline{Y}_C, \overline{R}_B$. we get

$$R_B = \frac{Q \cdot 2 + P_{2y} \cdot 5}{4} = \frac{5{,}6 \cdot 2 + 8{,}7 \cdot 5}{4} = 13{,}67\ kN,$$

From the first equation $X_C = P_{2x} = 5\ kN$,

From the second equation $Y_C = Q - R_B + P_{2y} = 5{,}6 - 13{,}67 + 8{,}7 = 0{,}63\ kN$,

From the fifth equation

$$Y_A = Y_C \sin 60° + X_C \sin 30° = 0{,}63 \cdot \frac{\sqrt{3}}{2} + 5 \cdot \frac{1}{2} = 3{,}05\ kN,$$

From the forth

$$M_A = -M + P_1 \cdot 2 - X_C \cos 30° \cdot 3 + Y_C \cos 60° \cdot 3 - X_C \sin 30° \cdot 2 -$$
$$-Y_C \sin 60° \cdot 2 = -15 + 6 \cdot 2 - 5\frac{\sqrt{3}}{2}3 + 0{,}63\frac{1}{2}3 - 5\frac{1}{2}2 - 0{,}63\frac{\sqrt{3}}{2}2 =$$
$$= -21{,}14\ kNm.$$

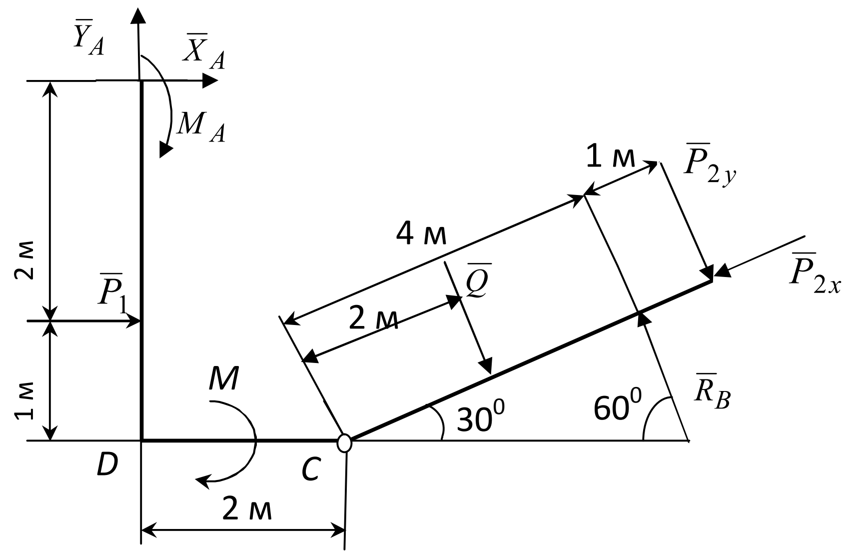
from the sixth $X_A = -Y_C \cos 60° + X_C \cos 30° - P_1 = -0,63 \cdot \frac{1}{2} + 5 \cdot \frac{\sqrt{3}}{2} - 6 = -1,98\, kN$

For checking we consider the construction as a whole and make for it an equation of moments about the point *C*, through which the lines of action of certain reactions do not pass. Pin *C* is considered solidified.

Checking:

$$\sum_{i=1}^{n} M_C(\overline{F}_i) = -M_A - X_A \cdot 3 - Y_A \cdot 2 - P_1 \cdot 1 - M - Q \cdot 2 + R_B \cdot 4 - P_{2y} \cdot 5 =$$

$$= -(-21{,}14) - (-1{,}98) \cdot 3 - 3{,}05 \cdot 2 - 6 \cdot 1 - 15 - 5{,}6 \cdot 2 + 13{,}67 \cdot 4 - 8{,}7 \cdot 5 \approx 0.$$

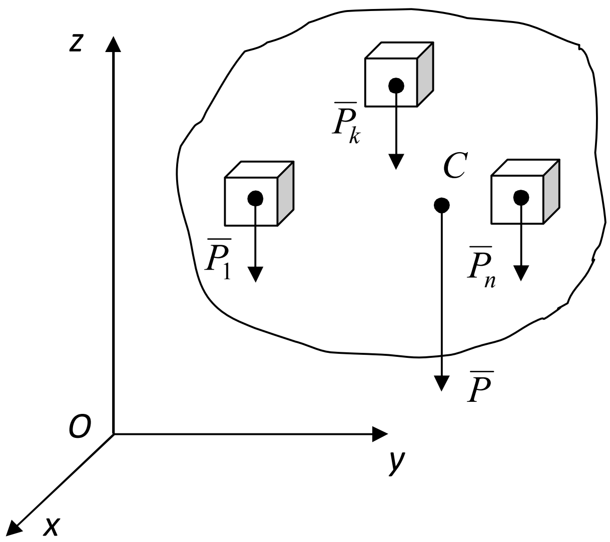


Picture 32

The sum moments of forces about the point *C* is zero means the problem is solved correctly.

8 CENTER OF GRAVITY OF A RIGID BODY

Consider the rigid body that is in the field of gravity. If the size of the body can be neglected in comparison with the size of the Earth, then we can assume that the particles of this body are forces of gravity $\overline{P}_k$, which composite a system of parallel forces.



Picture 33

The center of gravity of a rigid body is the center of parallel forces of gravity.

The coordinates of the center of gravity are determined by the formulas:

$$x_c = \frac{\sum_{k=1}^{n} x_k P_k}{P}, \quad y_c = \frac{\sum_{k=1}^{n} y_k P_k}{P},$$

$$z_c = \frac{\sum_{k=1}^{n} z_k P_k}{P},$$

where $P = \sum_{k=1}^{n} P_k$ – force of gravity; x_k, y_k, z_k – the coordinates of the point of gravity force $\overline{P}_k$ application of the body's parts.

8.1 The center of gravity of a rigid body

If the body is homogeneous, its *specific weight* ***γ*** is proportional to its volume:

$$P_k = \gamma_1 V_k,$$

where V_k is an infinitesimal volume, γ_1 is *specific weight*.

Substituting, we get:

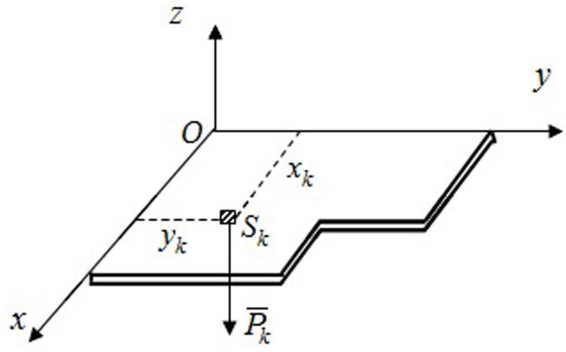
similarly

$$\left.\begin{aligned} x_c &= \frac{\sum_{k=1}^{n} x_k P_k}{P} = \frac{\sum_{k=1}^{n} x_k \gamma_1 V_k}{\gamma_1 V} = \frac{\gamma_1 \sum_{k=1}^{n} x_k V_k}{\gamma_1 V} = \frac{\sum_{k=1}^{n} x_k V_k}{V}, \\ y_c &= \frac{\sum_{k=1}^{n} y_k V_k}{V}, \\ z_c &= \frac{\sum_{k=1}^{n} z_k V_k}{V}, \end{aligned}\right\}$$

where $V = \sum_{k=1}^{n} V_k$ – is body's volume.

The formulas show that the position of the gravity center of a homogeneous body depends only on the geometric shape and size of the body. Therefore, the point C, which defined by the formulas, is ***the centroid of volume***.

8.2 The centroid of a surface



Picture 34

A plate is body, one which size (thickness) is much smaller than the other two (length and width):

$$P_k = \gamma_2 S_k ,$$

where S_k – is the area of the elemental surface; γ_2 – unit weight.

Here it supposed that the plate lies in the coordinate plane ***xOy*** so for expressions we get:

$$\left.\begin{aligned} x_c &= \frac{\sum_{k=1}^{n} x_k P_k}{P} = \frac{\sum_{k=1}^{n} x_k \gamma_2 S_k}{\gamma_2 S} = \frac{\gamma_2 \sum_{k=1}^{n} x_k S_k}{\gamma_2 S} = \frac{\sum_{k=1}^{n} x_k S_k}{S}, \\ y_c &= \frac{\sum_{k=1}^{n} y_k S_k}{S}, \end{aligned}\right\}$$

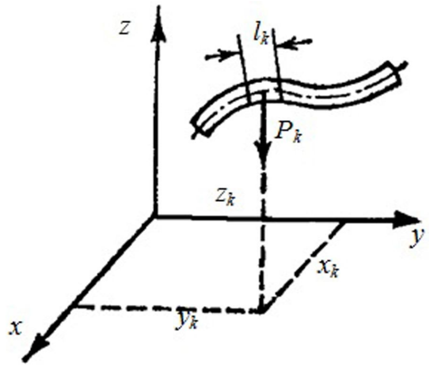
where $S = \sum_{k=1}^{n} S_k$ – is area of the plate.

The point *C* with coordinates x_c, y_c is the ***centroid of a surface***.

Expressions in numerators formulas are first moments of the area S_y and S_x with respect to the ***x*** and ***y*** axes:

$$S_y = \sum_{k=1}^{n} x_k S_k = x_c \cdot S, \quad S_x = \sum_{k=1}^{n} y_k S_k = y_c \cdot S.$$

8.3 The centroid of a line



Picture 35

Uniform rod is the body, which one sizes (*the length*) is substantially greater than the other two dimensions. In this case, the unit weight is proportional to its length

$$P_k = \gamma_3 l_k ,$$

де l_k – unit length of the rod; γ_3 – linear weigh.

Substituting, we get:

$$x_c = \frac{\sum_{k=1}^{n} x_k P_k}{P} = \frac{\sum_{k=1}^{n} x_k \gamma_3 l_k}{\gamma_3 L} = \frac{\gamma_3 \sum_{k=1}^{n} x_k l_k}{\gamma_3 L} = \frac{\sum_{k=1}^{n} x_k l_k}{L},$$

similarly
$$y_c = \frac{\sum_{k=1}^{n} y_k l_k}{L},$$

$$z_c = \frac{\sum_{k=1}^{n} z_k l_k}{L},$$

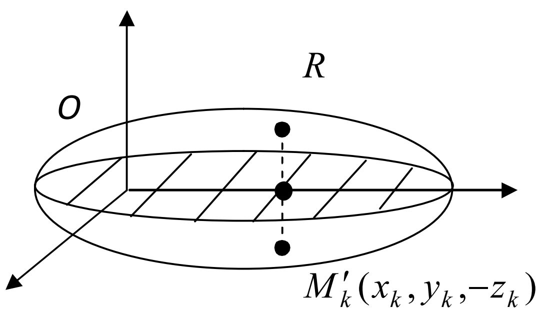
where $L = \sum_{k=1}^{n} l_k$ is the rod length.

The point C with coordinates x_c, y_c, z_c is the ***centroid of a line***.

8.4 Methods to determine the coordinates of the centroid

1. ***Method of symmetry***. If a body is uniform and symmetric with respect to a plane, the body's centroid lies in this plane.

Let us prove this statement for a body having a plane of symmetry. The position the coordinate plane xOy is in the plane of symmetry (in the figure this plane is shaded).



Picture 36

Let's take two points in the body M_k and M'_k which are symmetric relative to the plane xOy. These points coincide coordinates x_k, y_k, and coordinate z_k differ only in the sign. We select around points M_k, M'_k equals elemental volumes V_k.

To summarize applications: $z_k V_k + z'_k V_k = z_k V_k - z_k V_k = 0$.

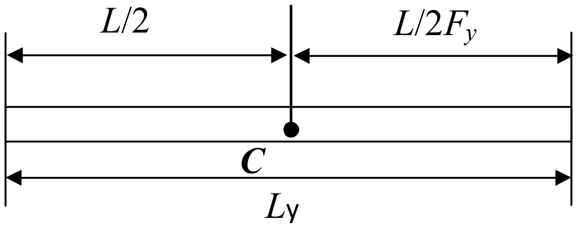
If considered all elementary volumes, we get $\sum_{k=1}^{n} z_k V_k = 0$

and calculate the coordinate of the centroid z_c for the formula $z_c = \frac{\sum_{k=1}^{n} z_k V_k}{V} = 0$.

This means that the centroid of the considered body is in the plane of symmetry.
Similarly, we can prove the statement for a body that has an axis or center of symmetry.

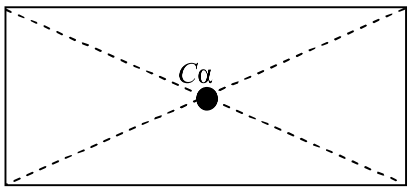
Problems. Considering some problems.

a) straight rod



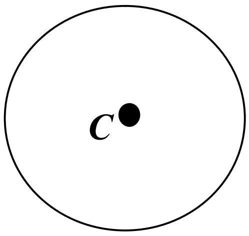
The center of symmetry of a rod is a point in the middle of the rod. The center of gravity of a straight rod – the point C – is in the middle of the rod;

б) rectangle



The center of symmetry of a rectangle is the point of intersection of its diagonals. The centroid of the rectangle – the point C – is also at the intersection of the diagonals. As you know, the diagonals at the intersection divided in half;

в) circle.

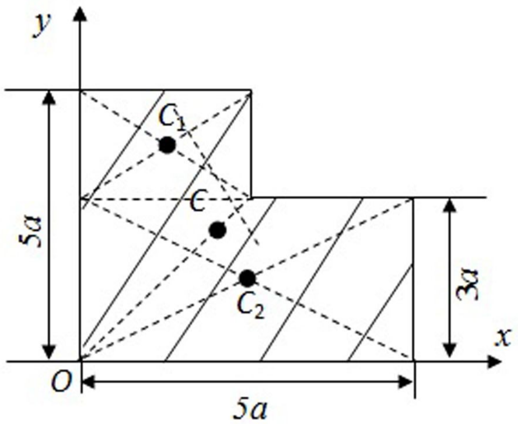


The center of symmetry, and mean *the centroid of the circle*, is its center.

2. ***Method of dividing body.***

If the body can divided into a finite number of parts for which the positions of the centroid are known, then the coordinates of the center of gravity can be calculated by the formulas.

Problem 1. Determine the coordinates of the surface centroid.



Picture 37

Solution. Let divide the surface into two rectangles which centers of gravity C_1 and C_2 are at the points of the diagonals intersection. We will choose a coordinate system Oxy. Data on the coordinates of centroids of rectangles and their area will written in the table.

Table 1

k	x_k	y_k	S_k
1	1,5a	4a	6a^2
2	2,5a	1,5a	15a^2

The coordinates of the centroid of the surface found by the formulas:

$$x_c = \frac{\sum_{k=1}^{n} x_k S_k}{S} = \frac{x_1 S_1 + x_2 S_2}{S_1 + S_2} = \frac{1{,}5a \cdot 6a^2 + 2{,}5a \cdot 15a^2}{6a^2 + 15a^2} = \frac{9a^3 + 37{,}5a^3}{21a^2} =$$

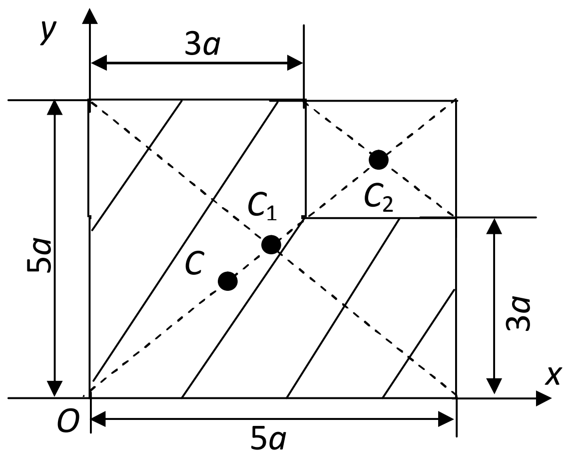
$$= \frac{46{,}5}{21} a \approx 2{,}2a,$$

$$y_c = \frac{\sum_{k=1}^{n} y_k S_k}{S} = \frac{y_1 S_1 + y_2 S_2}{S_1 + S_2} = \frac{4a \cdot 6a^2 + 1{,}5a \cdot 15a^2}{6a^2 + 15a^2} = \frac{24a^3 + 22{,}5a^3}{21a^2} = \frac{46{,}5}{21} a \approx 2{,}2a.$$

The values of the point coordinates C (2,2a, 2,2a) indicate that it lies on the angle bisector taken from the center of coordinates, which is the line of symmetry of the plane.

3. ***Method of negative areas***. If the body has a hole (cutout), then this hole (cutout) can considered as a body with a negative area (weight) and for calculations use the method of dividing.

Problem 1. Let's consider the problem of Example 1.



Picture 38

Solution. Imagine the area as a square with the sides $5a \times 5a$ and the center of gravity C_1, from which the square was cut with the sides $2a \times 2a$ and the center of gravity C_2. The area of the last square will considered negative. Data on the coordinates of centroids of rectangles and their area will written in the table.

Table 2

k	x_k	y_k	S_k
1	2,5a	2,5a	25a^2
2	4a	4a	-4a^2

The coordinates of the centroid of the surface can found by the formulas:

$$x_c = \frac{\sum\limits_{k=1}^{n} x_k S_k}{S} = \frac{x_1 S_1 + x_2 S_2}{S_1 + S_2} = \frac{2{,}5a \cdot 25a^2 + 4a \cdot (-4a^2)}{25a^2 + (-4a^2)} = \frac{62a^3 - 16a^3}{25a^2 - 4a^2} = \frac{46{,}5}{21} a \approx 2{,}2a;$$

$$y_c = \frac{\sum\limits_{k=1}^{n} y_k S_k}{S} = \frac{y_1 S_1 + y_2 S_2}{S_1 + S_2} = \frac{2{,}5a \cdot 25a^2 + 4a \cdot (-4a^2)}{25a^2 + (-4a^2)} = \frac{62{,}5a^3 - 16a^3}{25a^2 - 4a^2} = \frac{46{,}5}{21} a \approx 2{,}2a.$$

4. ***Method of integration***. If the body cannot divided into a finite number of parts, the formulas used to the integral.

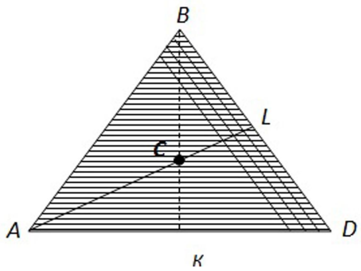
$$\left.\begin{aligned} x_c &= \frac{\int\limits_{(S)} x dS}{S} \\ y_c &= \frac{\int\limits_{(S)} y dS}{S} \end{aligned}\right\},$$

where integrals extend to the area.

8.5 Centroid location simplest figure

Several simplest figures, from which more complex figures can be formed.

a) triangle.

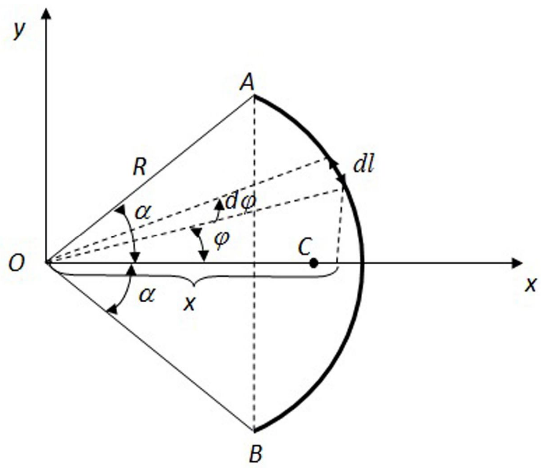


Picture 39

The centroid of a triangle is at the intersection of median. This point divides each of the medians in relation to 1:2, i.e.

$CK : CB = 1:2$, $CL : CA = 1:2$;

b) circle arc segment.



Picture 40

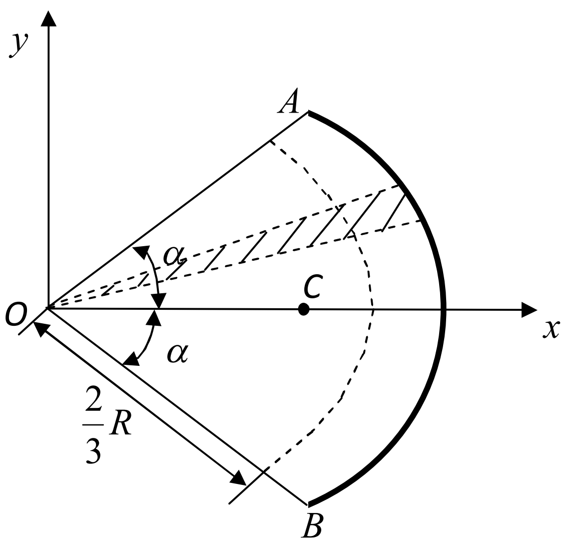
Consider the arc of AB circle with radius R and a central angle. We direct the Ox axis along the axis of symmetry of the arc, which is a bisector of the angle 2α . The centroid of the circle arc lies on the axis of symmetry, i. e., and remains to found. To use the formula

$$x_C = \frac{\int_A^B x dl}{L}.$$

For an unit length, as in Figure, $x = R\cos\varphi$, $dl = Rd\varphi$, $L = R \cdot 2\alpha$. So

$$x_C = \frac{\int_{-\alpha}^{\alpha} R\cos\varphi R d\varphi}{R\cdot 2\alpha} = \frac{R^2\int_{-\alpha}^{\alpha}\cos\varphi d\varphi}{R\cdot 2\alpha} = \frac{R^2 \sin\alpha\Big|_{-\alpha}^{\alpha}}{R\cdot 2\alpha} = \frac{R^2 2\sin\alpha}{R\cdot 2\alpha} = \frac{R\sin\alpha}{\alpha}$$

c) circular sector area.



Picture 41

The geometric point of the centroids of all unit triangles will be the arc of a radius circle $\frac{2}{3}R$. In this case, you can use the formula for the centroid of the circle arc:

$$x_c = \frac{2}{3}R\frac{\sin\alpha}{\alpha}$$

Note. In the formulas, the angle must take in *radians*.

REFERENCES

1. Теоретична механіка : підручник / І. В. Кузьо, В. П. Шпачук, Н. М. Ванькович та ін. – Харків : Фоліо, 2017. – 780 с.

2. Теоретична механіка. Статика : конспект лекцій для студентів денної і заочної форм навчання бакалаврів за спеціальностями 192 – Будівництво та цивільна інженерія, 141 – Електроенергетика, електротехніка та електромеханіка, 263 – Цивільна безпека, 275 – Транспортні технології (за видами), 185 – Нафтогазова інженерія та технології) / В. П. Шпачук, О. І. Рубаненко, А. О. Гарбуз ; Харків. нац. ун-т міськ. госп-ва ім. О. М. Бекетова. – Харків : ХНУМГ ім. О. М. Бекетова, 2017. – 125 с.

3. Теоретична механіка : навч.-метод. посібник і завдання для контрольних і самостійних робіт / за заг. ред. В. П. Шпачука. – Харків: ХНУМГ, 2015. – 146 с.

4. Павловський М. А. Теоретична механіка : Підручник / М. А. Павловський. – Київ : Техніка, 2002. – 512 с.

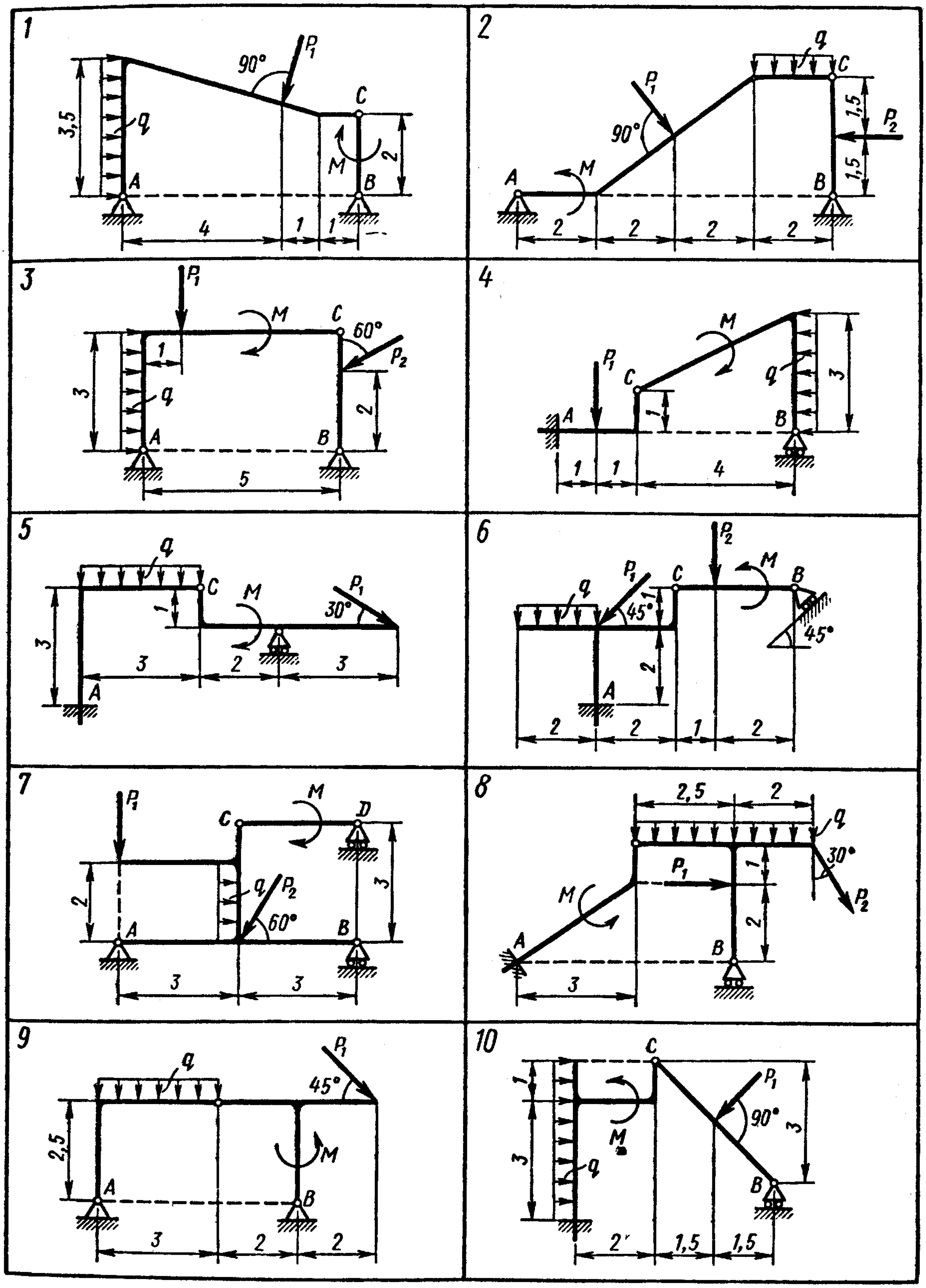
APPENDIX

Output dates for work

"Determination of the reactions of the supports of the body system"

The structure consists of two bodies. Determine the reaction of the constrains *A* and *B*, as well as the pressure in the intermediate hinge *C*. The method of solving the problems considered in Section 7**.**

Number of variant	P_1	P_2	M, kN·m	q, kN/m
	kN			
1	5,0	–	24,0	0,8
2	6,0	10,0	22,0	1,0
3	7,0	9,0	20,0	1,2
4	8,0	–	18,0	1,4
5	9,0	–	16,0	1,6
6	10,0	8,0	25,0	1,8
7	11,0	7,0	20,0	2,0
8	12,0	6,0	15,0	2,2
9	13,0	–	10,0	2,4
10	14,0	–	12,0	2,6
11	15,0	5,0	14,0	2,8
12	12,0	4,0	16,0	3,0
13	9,0	6,0	18,0	3,2
14	6,0	–	20,0	3,4
15	5,0	8,0	22,0	3,6
16	7,0	10,0	14,0	3,8
17	9,0	12,0	26,0	4,0
18	11,0	10,0	18,0	3,5
19	13,0	9,0	30,0	3,0
20	15,0	8,0	25,0	2,5
21	10,0	7,0	20,0	2,0
22	5,0	6,0	15,0	1,5
23	8,0	5,0	10,0	1,4
24	11,0	4,0	5,0	1,3
25	14,0	6,0	7,0	1,2
26	12,0	8,0	9,0	1,1
27	10,0	7,0	11,0	1,0
28	8,0	9,0	13,0	1,2
29	5,0	7,0	10,0	1,5
30	10,0	12,0	17,0	1,6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90°
P_1
C
3,5
q
M
2
A
B
4
1
1
2
q
C
P_1
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P_2
90°
M
A
B
1,5
2
2
2
2
3
P_1
M
C
60°
P_2
1
3
q
2
A
B
5
4
M
P_1
C
q
3
1
A
B
1
1
4
5
q
C
M
P_1
30°
1
3
3
2
3
A
6
P_2
P_1
C
M
B
q
45°
1
45°
2
A
2
2
1
2
7
P_1
M
C
D
q
P_2
2
3
60°
A
B
3
3
8
2,5
2
q
30°
P_1
1
M
P_2
2
A
B
3
9
q
P_1
45°
2,5
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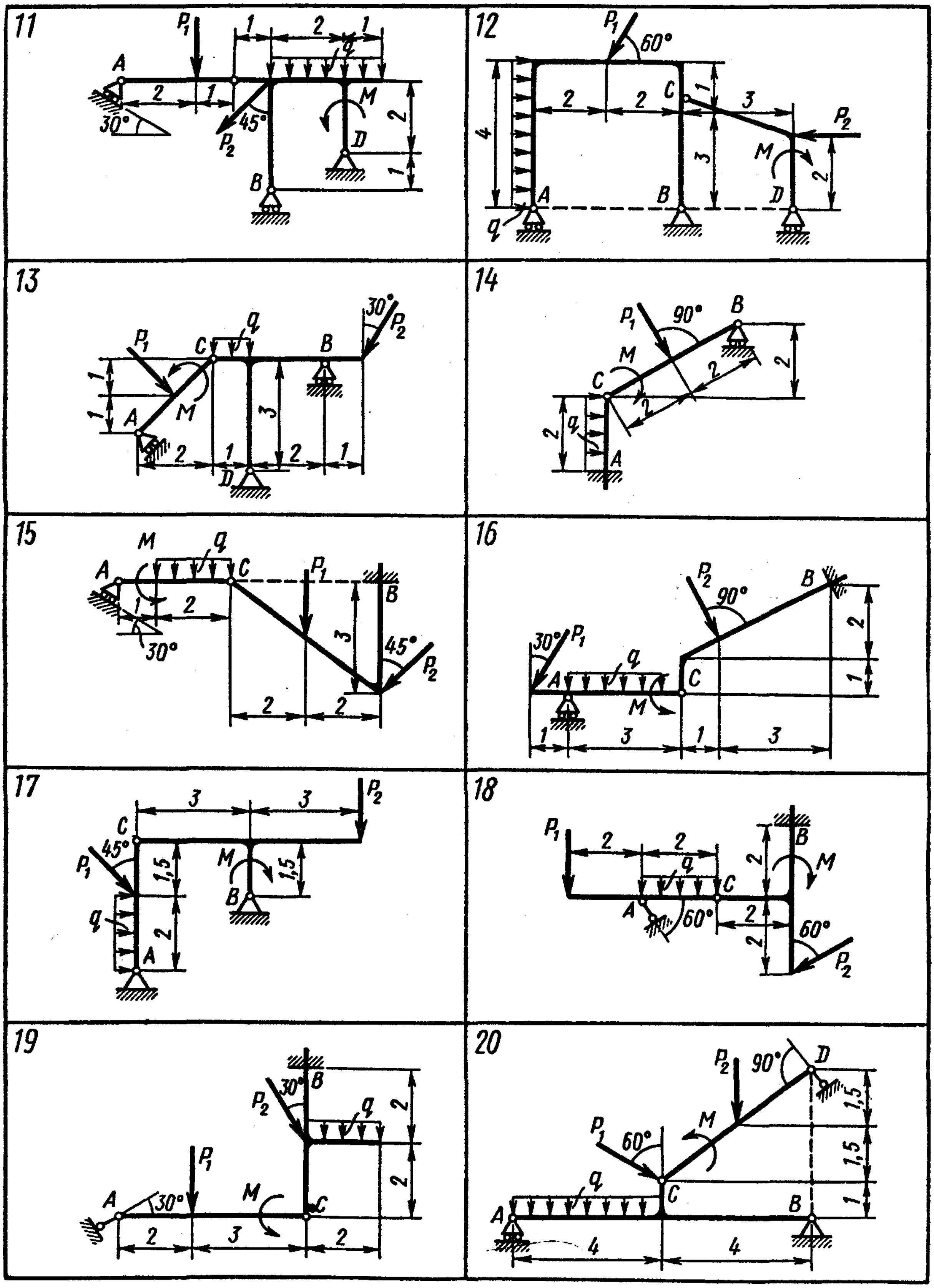
1,5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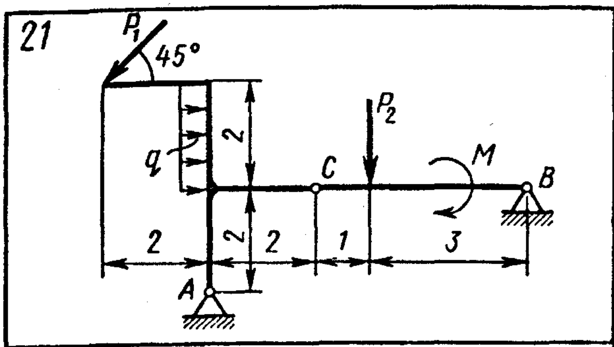

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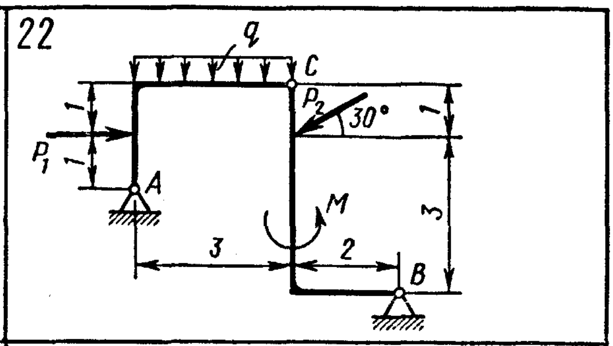

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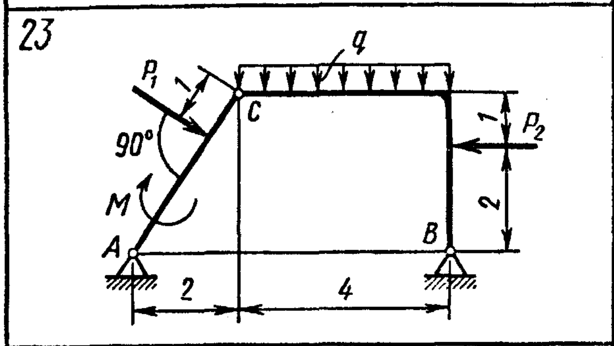

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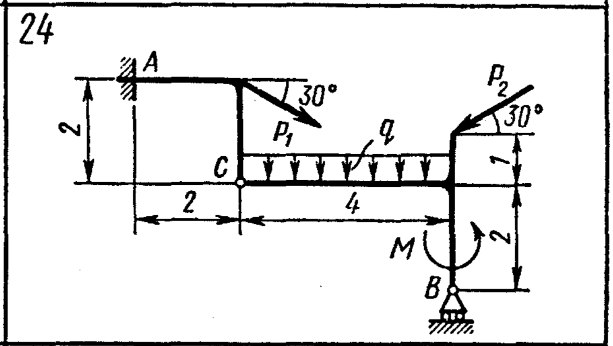

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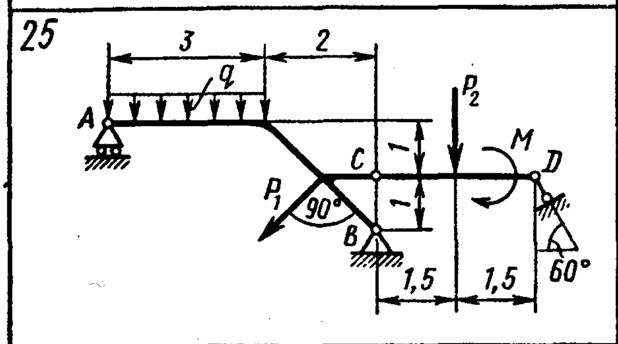

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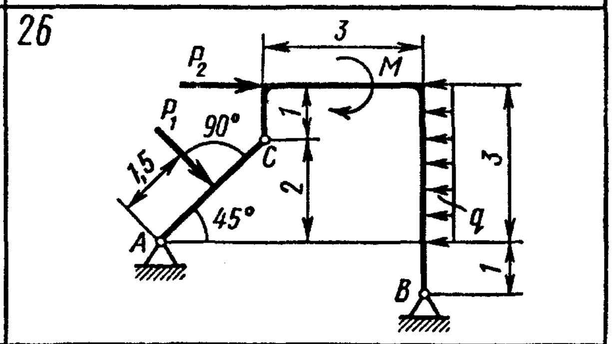

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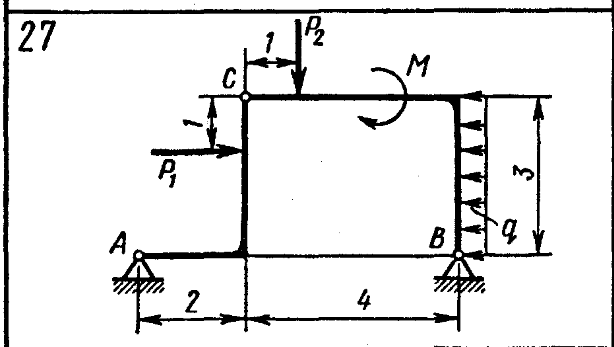

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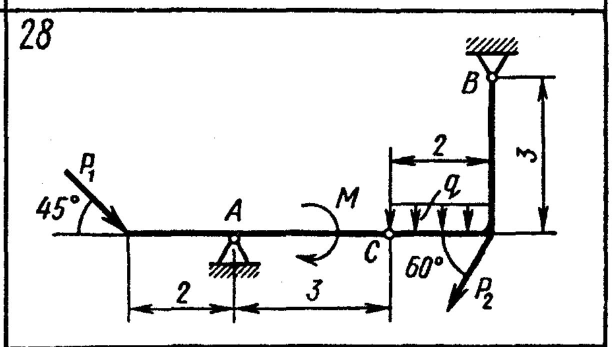

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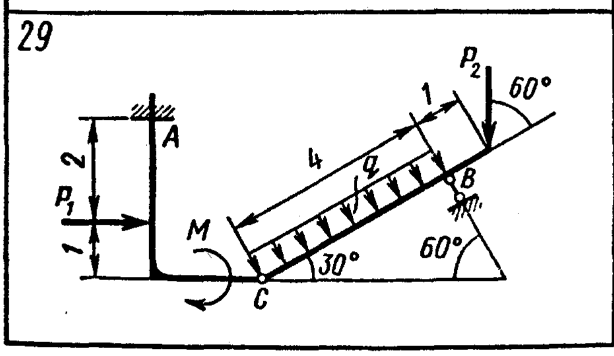

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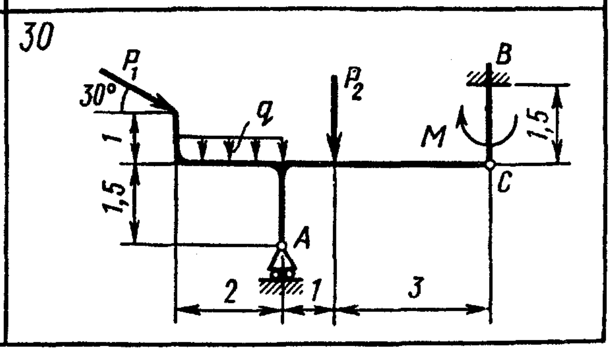

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Методичні рекомендації
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та розрахунково-графічних робіт
з навчальної дисципліни

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(Частина 1 Статика)

(англ. мовою)
(*для студентів-бакалаврів 1 курсу денної форми навчання*
за спеціальністю
192 – Промислове та цивільне будівництво)

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